Turbulence in small-world networks

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The transition to turbulence via spatiotemporal intermittency is investigated in the context of coupled maps defined on small-world networks. The local dynamics is given by the Chaté-Manneville minimal map previously used in studies of spatiotemporal intermittency in ordered lattice. The critical boundary separating laminar and turbulent regimes is calculated on the parameter space of the system, given by the coupling strength and the rewiring probability of the network. Windows of relaminarization are present in some regions of the parameter space. New features arise in small-world networks; for instance, the character of the transition to turbulence changes from second-order to a first-order phase transition at some critical value of the rewiring probability. A linear relation characterizing the change in the order of the phase transition is found. The global quantity used as order parameter for the transition also exhibits nontrivial collective behavior for some values of the parameters. These models may describe several processes occurring in nonuniform media where the degree of disorder can be continuously varied through a parameter.

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I. INTRODUCTION

A general scenario for the appearance of turbulence in extended systems is spatiotemporal intermittency, i.e., a sustained regime characterized by the coexistence of coherent-laminar and disordered-chaotic domains in space and time [1,2]. Some of the most extensive investigations of this phenomenon have been based on model dynamical systems such as coupled map lattices [2–6]. The idea is that the ingredients of such models—a discrete space, discrete time system of interacting elements whose states vary continuously according to specific functions—are sufficient to capture much of the phenomenology observed in complex spatiotemporal processes, in particular some relevant features of spatiotemporal intermittency and turbulence. In this respect, coupled map systems can be considered as mathematically simpler and computationally more efficient models than partial differential equations of hydrodynamics [7]. The transition to turbulence via spatiotemporal intermittency has mainly been investigated in networks of coupled maps where the connectivities between elements are defined from deterministic rules that provide order to their spatial structure. There are examples of such studies in Euclidean lattices [2–6], fractal geometries [8], and treelike arrays [9], as well as on globally coupled systems [10]. These investigations have allowed the characterization of the transition to turbulence as a critical phenomenon in a range of Euclidean and fractal dimensions and on several connection topologies.

Because of their discrete spatial nature, coupled map systems seem especially appropriate for investigating physical phenomena occurring in heterogeneous or disordered media. Recently, intensive and interesting research has been performed on the theory and applications of small-world networks [11–14]. Small-world networks are a class of networks with a high degree of local clustering and a small characteristic length between any two elements. It has been shown that small-world networks describe many natural and artificial networks [15]. By varying a parameter, small-world networks can be continuously tuned between ordered, deterministic lattices and completely random networks. In this paper, we consider coupled maps defined on small-world networks as spatiotemporal dynamical systems. We study the nature of the transition to turbulence and the properties of spatiotemporal intermittency on these networks. We explore the changes induced in those processes as a result of the variation in the connection topology of the interactions in the system.

In Sec. II, a general coupled map lattice model for the treatment of small-world networks is presented. The transition to turbulence in coupled maps on small-world networks is investigated in Sec. III. The site map model that we employ is based on the one introduced earlier by Chaté and Manneville for regular Euclidean lattices in one and two dimensions [3], and which captures the essential features of the transition to turbulence in extended systems. Section IV contains the observations of nontrivial collective behavior arising in the system. Conclusions are given in Sec. V.

II. COUPLED MAPS ON SMALL-WORLD NETWORKS

There are several ways to construct a small-world network. In this paper we employ the small-world network construction algorithm originally proposed by Watts and Strogatz [11]. We start from a ring of N sites, where each site is connected to its k nearest neighbors, k being an even number. Then each connection is rewired at random with probability p to any other site of the network, to avoid self-connections. After the rewiring process, the number of elements coupled to each site (which we call neighbors of that site) may vary, but the total number of links in the network is constant and equal to $Nk/2$. It is assumed that all links are bidirectional. Although this algorithm does not guarantee that the resulting

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graph is connected, we have used only connected ones for our calculations.

The state of each site on the network can be assigned a continuous variable, which evolves according to a deterministic rule depending on its own value and the values of its connecting elements. We define a coupled map lattice dynamical system on a small-world network as

$$x_{t+1}(i)=f[x_t(i)]+\frac{\epsilon}{k(i)}\sum_{j\in S(i)}\{f[x_t(j)]-f[x_t(i)]\}, \tag{1}$$

where $x_t(i)$ gives the state of the site i $(i=1,2,\ldots,N)$ at discrete time t; $k(i)$ is the number of neighbors or elements connected to site i; $S(i)$ is the set of neighbors of site i; ϵ is a parameter measuring the coupling strength between connected sites, and $f(x)$ is a nonlinear function describing the local dynamics.

The above coupled map equations can be generalized to include other coupling schemes or continuous time local dynamics. Different spatiotemporal processes can be studied on small-world structures by providing appropriate local dynamics and couplings.

III. TRANSITION TO TURBULENCE IN SMALL-WORLD NETWORKS

The phenomenon of spatiotemporal intermittency in extended systems consists of a sustained regime where coherent and chaotic domains coexist and evolve in space and time. A local map possessing the minimal requirements for observing spatiotemporal intermittency in coupled map systems is [3]

$$f(x)=\begin{cases}\frac{r}{2}(1-|1-2x|), & \text{if } x\in[0,1]\\ x, & \text{if } x>1,\end{cases} \tag{2}$$

with $r>2$. This map is chaotic for $f(x)$ in $[0,1]$. However, for $f(x)>1$ the iteration is locked on a fixed point. The local state can thus be seen as a continuum of stable ''laminar'' fixed points $(x>1)$ adjacent to a chaotic repeller or ''turbulent'' state $(x\in[0,1])$.

In ordered, deterministic networks, the turbulent state can propagate through the lattice in time for a large enough coupling, producing sustained regimes of spatiotemporal intermittency. Here, we investigate the phenomenon of transition to turbulence in small-world networks using the local map f [Eq. (2)] in the coupled system described by Eq. (1). The local parameter is fixed at the value $r=3$ in all the calculations. As in ordered networks, the transition to the extended turbulent state can be characterized through the average value of the instantaneous fraction of turbulent sites F_t, a quantity that acts as the order parameter for the system [3]. We have calculated $\langle F\rangle$ as a function of various parameters of the system from a time average of the instantaneous turbulent fraction F_t as

$$\langle F\rangle=\frac{1}{T}\sum_{t=1}^{T}F_t. \tag{3}$$

Thus $\langle F\rangle=0$ describes a laminar state and $\langle F\rangle\in(0,1]$ corresponds to a turbulent state of the network. Random initial conditions were used in all runs with given parameter values of the system (1). About 10^4 iterations were discarded before taking the time average in Eq. (3) and T was typically taken at the value 10^4. Increasing the averaging time T or the network size N does not have appreciable effects on the results. It should be noticed that a minimum number of initially turbulent sites is always required to reach a sustained state of turbulence.

Figure 1 shows $\langle F\rangle$ as a function of ϵ for different fixed values of the probability p in a typical small-world network with nearest neighbor number $k=10$ and size $N=10^4$. The error bars shown on $\langle F\rangle$ correspond to plus and minus the standard deviations $\Delta\langle F\rangle$ (square root of the variance) of the time series of F_t at each value of the coupling parameter ϵ. We have verified, by doing 25 realizations of the rewiring process described in Sec. I, that fluctuations on $\langle F\rangle$ and on $\Delta\langle F\rangle$ due to different configurations of the networks are not significant.

Figures 1(a)–(d) show that, for a fixed value of p, there is a critical value of the coupling ϵ_c at which the transition from a laminar state to turbulence occurs. For small values of p [Fig. 1(a)], a regime of relaminarization of the system takes place on an interval of the coupling parameter after a window of turbulence. The relaminarization gap disappears with increasing p, leaving a dip in the $\langle F\rangle$ curve, as seen in Fig. 1(b). For small values of p the transition to turbulence, as the coupling is varied, takes place continuously, similarly to a second-order phase transition. As p is increased, the transition becomes progressively steeper until it happens discontinuously, as in a first-order phase transition. There exists a critical value of the probability $p_c\approx0.55$ at which the character of the transition to turbulence changes from a second-order phase transition [Figs. 1(a) and 1(b)] to a first-order phase transition [Figs. 1(c) and 1(d)].

In Figs. 2(a) and 2 (b) we show the mean turbulent fraction $\langle F\rangle$ and its standard deviation $\Delta\langle F\rangle$ plotted as functions of both p and ϵ. Figure 2(a) shows that the transition to sustained turbulence occurs on a critical curve on the parameter plane (p,ϵ). The variation in the nature of this transition along this critical curve as p increases can clearly be appreciated in Fig. 2(a). Typical statistical deviations are seen for small values of p and ϵ in Fig. 2(b); however, this figure reveals very large fluctuations in the instantaneous turbulent fraction occurring for larger values of those parameters, and which can also be observed as the ''bulbs'' in Figs. 1(c) and 1(d). As it shall be discussed in the following section, this phenomenon is associated with the emergence of nontrivial collective behavior in the system.

We have numerically calculated the critical values of the coupling ϵ_c for the onset of turbulence in small-world networks with fixed $k=10$ as a function of their rewiring probability p. Figure 3 shows the resulting critical boundary $\epsilon_c(p)$ for the transition to turbulence, as well as the phase

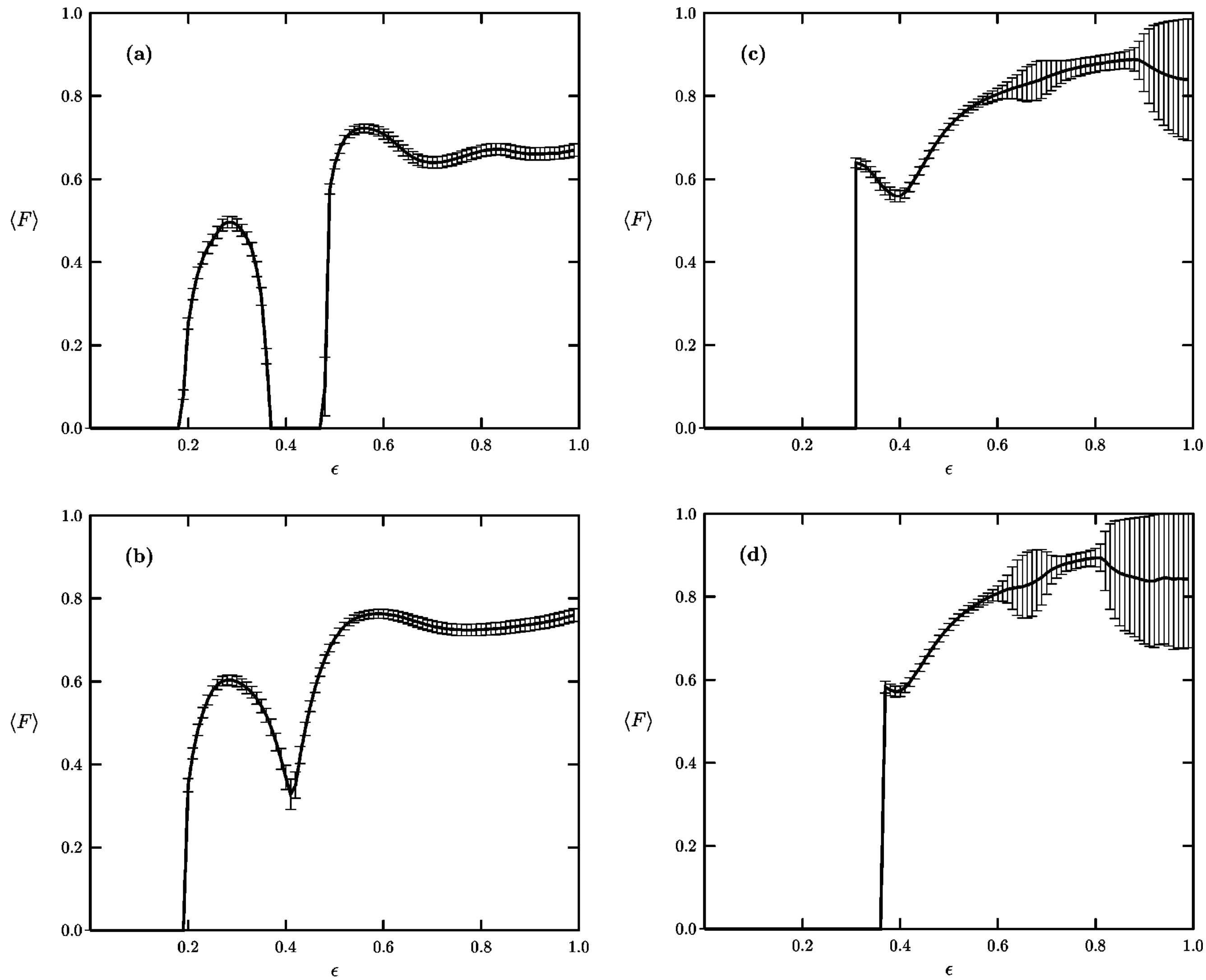

FIG. 1. Mean turbulent fraction $\langle F\rangle$ as a function of the coupling ϵ for small-world networks with $k=10$ and $N=10^4$. The error bars indicate ± 1 standard deviations. (a) $p=0$, (b) $p=0.17$, (c) $p=p_c=0.55$, and (d) $p=0.80$.

diagram of the system, on the parameter plane (p,ϵ). The critical coupling value ϵ_c increases as the disorder in the network, described by the probability p, grows. The critical boundary curve $\epsilon_c(p)$ corresponds to a continuous, second-order phase transition for $p<p_c=0.55$ and to a discontinuous, first-order transition for $p>p_c$. Figure 3 indicates with a dotted line where the first maximum of the mean turbulent fraction $\langle F\rangle$ occurs on the parameter plane. This line of first maxima of $\langle F\rangle$ crosses the critical boundary separating the laminar and the turbulent states of the system at the value $p=p_c$, and the character of the phase transition changes at this point. The relaminarization gap is also indicated on the plane (p,ϵ).

For values of the probability $p<p_c$, where a continuous transition from a laminar regime to turbulence occurs, the variation of the order parameter $\langle F\rangle$ near the critical curve can be characterized by a critical exponent β as $\langle F\rangle\sim(\epsilon-\epsilon_c)^\beta$. For fixed p, the exponent β can be calculated from a log-log plot of $\langle F\rangle$ vs $(\epsilon-\epsilon_c)$. The critical exponent β varies continuously with the rewiring probability p. In Fig. 4 we show the resulting graphs of β as a function of p for several small-world networks characterized by their neighbor number k. In each case, the dependence of the exponent β with p is well accounted by the linear relation $\beta=h(p_c-p)$, where the slope h varies with k. As p increases, the exponent β becomes smaller and the corresponding phase transition from laminarity to turbulence gets more abrupt. The change in the character of the transition from second order to first order should occur at the value $p=p_c$ for which the exponent β vanishes. Figure 4 shows the extrapolation of the straight line corresponding to the small-world network with $k=10$ until its intersection with the p axis, predicting a critical value $p_c=0.55$. This, in fact, is the critical value of the probability at which the change in the character of the transition to turbulence for this network was observed in Figs. 1(c), 2(a), and 3. We recall that the measure of the characteristic path length for the family of small-world networks with $k=10$ drops to small values typically associated with a random network when the rewiring probability is about 0.55

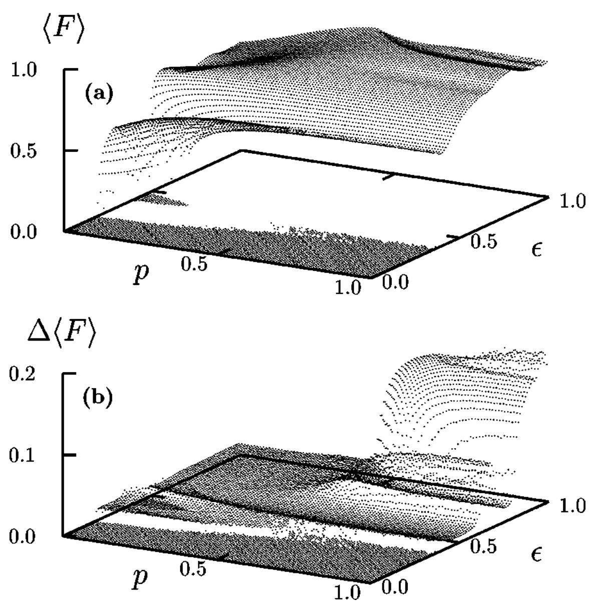

FIG. 2. (a) Mean turbulent fraction $\langle F \rangle$ as a function of p and ϵ. (b) Standard deviation $\Delta\langle F \rangle$ as a function of p and ϵ. Network parameters are $k=10$ and $N=10^4$.

[11]. Thus the critical value $p_c = 0.55$ found for the emergence of the first order phase transition should be related to the onset of randomness in the network. In fact, it has recently been reported that the transition from laminarity to turbulence in randomly coupled map networks can occur discontinuously [16].

By extrapolating the different lines, one can get predictions of the critical probabilities p_c for different values of k. We have numerically verified these critical values p_c for several small-world networks possessing different neighbor numbers k. Thus the linear relation that arises from Fig. 4

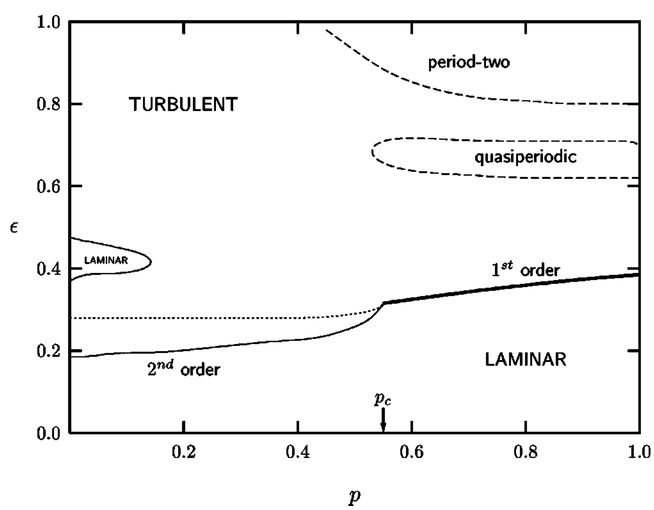

FIG. 3. Phase diagram of the system (1). The critical boundary separating laminar and turbulent regimes is shown with a continuous line. Thin line: second-order phase transition; thick line: first-order phase transition. The dotted line indicates the locus of the first maxima of $\langle F \rangle$ on the parameter plane. The regions of nontrivial collective behavior are bounded by dashed lines.

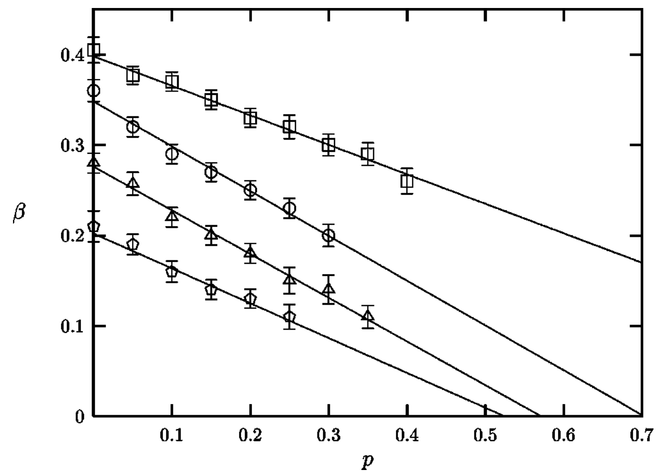

FIG. 4. Critical exponent β for the second-order phase transtion as a function of the rewiring probability p. $k=4$ (squares); $k=6$ (circles); $k=10$ (triangles); and $k=20$ (pentagons).

seems to characterize the transition to turbulence via spatiotemporal intermittency in small-world networks. Furthermore, Fig. 4 predicts that no critical value $p_c \leqslant 1$ can be found in the case $k=4$, and thus the transition to turbulence in a small-world network with neighbor number $k=4$ should occur continuously (second order) for any p, and should always possess a critical exponent $\beta > 0$. This prediction was also verified numerically.

Figure 5 shows the predicted critical probability values p_c as a function of k. Note that p_c decreases rapidly with increasing neighbor number k. The characteristic path length becomes smaller with increasing k [11] and, consequently, the connectivity of the small-world network approaches the all-to-all coupling limit of a globally coupled system, where the transition to turbulence is always a first-order phase transition [10]. In globally coupled systems, the absence of spatial relations excludes the possibility of supporting small domains of turbulent maps that would be necessary for a continuous transition to turbulence. The decrease in the value of p_c observed in Fig. 5 means that the networks need less disorder for achieving a first-order phase transition to turbu-

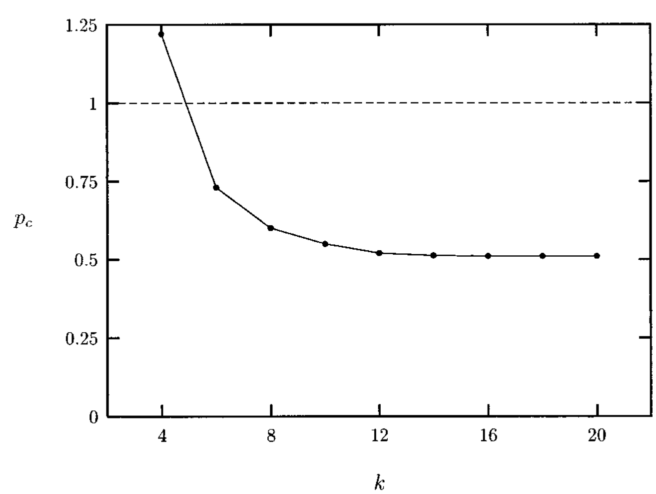

FIG. 5. Critical values of the rewiring probability vs k.

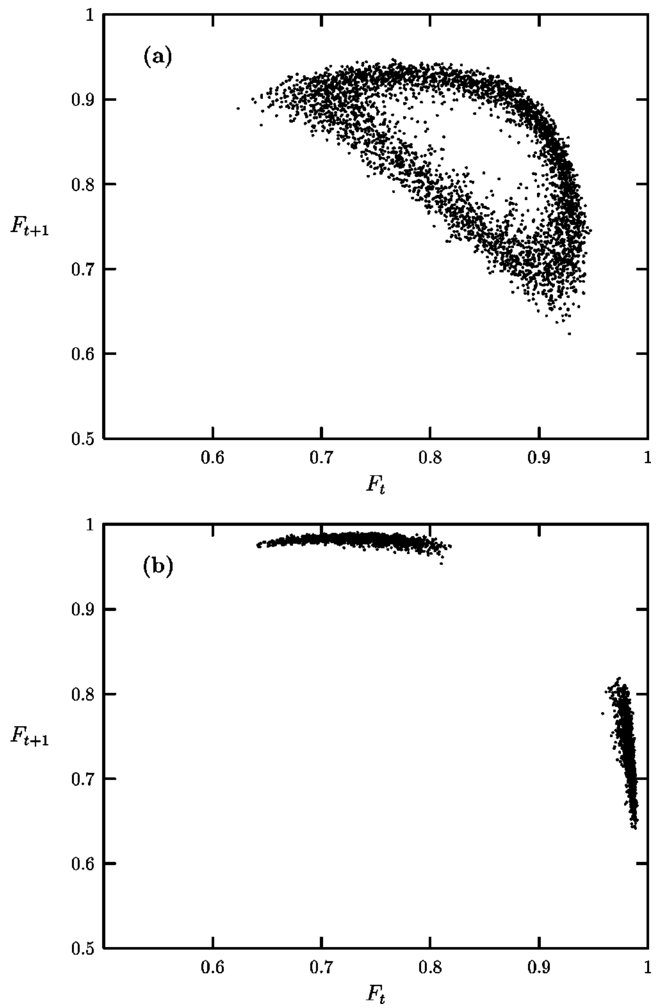

FIG. 6. Return maps of the instantaneous turbulent fraction for a small-world network with $k=10$, $N=10^4$, and $p=0.80$. (a) $\epsilon=0.67$ and (b) $\epsilon=0.85$.

lence if their neighbor number k is sufficiently large. This reflects the fact that networks with large enough k are already closer to global coupling, and therefore they require less rewiring to behave statistically similar to globally coupled systems.

IV. NONTRIVIAL COLLECTIVE BEHAVIOR

In contrast to the usually expected statistical behavior, the large amplitudes of the standard deviations observed on some regions of the parameter plane (p,ϵ) in Fig. 2(b) do not diminish with increasing system size or with longer averaging time. These large fluctuations of a statistical quantity indicate the presence of collective motions of the system. For example, Fig. 6 displays the return maps of the instantaneous turbulent fraction F_t for two different values of the coupling corresponding to the two "bulbs" observed in Fig. 1(d). Figure 6(a) shows a quasiperiodic orbit in the dynamics of F_t in the first bulb, while Fig. 6(b) reveals a collective period-two motion occurring in the second bulb of Fig. 1(d). The large fluctuations measured by $\Delta\langle F\rangle$ reflect the amplitude of the collective oscillations of F_t that emerge in those regions of parameters. This feature corresponds to a phenomenon of nontrivial collective behavior already observed in the temporal evolution of the mean field of several chaotic extended systems [17]. The quantity F_t is a simpler statistical description than the mean field and, in this case, it already manifests a nontrivial collective behavior of the system over long times.

The two regions on the parameter plane where states of nontrivial collective behavior appear are indicated within the turbulent zone in the phase diagram of Fig. 3. Note that the emergence of nontrivial collective behavior occurs for large enough coupling and an appreciable degree of disorder, which itself implies a small characteristic length between any two elements in the network. Thus, as both the amount of short cuts and the coupling strength between different parts of the network increase, the information transfer required for the emergence of collective behavior is more likely to occur.

Nontrivial collective oscillations in the turbulent phase of coupled map systems have also been observed in high-dimensional Euclidean lattices [18], fractal lattices [8], as well as in globally coupled maps [10]. Our results for small-world networks show that ordered connections are not essential for the occurrence of nontrivial collective behavior.

V. CONCLUSIONS

We have investigated a coupled map model for the transition to turbulence via spatiotemporal intermittency in small-world networks. Although the local dynamics is simple, we expect that the essential properties of the transition to turbulence in small-world structures is captured by this model. Coupled Chaté-Manneville maps could be regarded as a crude description of the dynamics of an excitable medium. The system of coupled maps on small-world networks can also be used to study different spatiotemporal dynamical processes on these structures by providing appropriate local maps or couplings.

By varying the rewiring probability in small-world networks, the behavior of the transition to turbulence can be studied in the regime between ordered lattices and completely random networks. The critical boundary separating the laminar and the turbulent regimes was calculated on the parameter plane (p,ϵ) of the system. We have found that the character of this transition changes progressively from a second-order phase transition to a first-order phase transition as the disorder in the network, measured by the rewiring probability, is increased. The critical value of the rewiring probability for the onset of the first-order phase transition was predicted from the scaling behavior observed in the critical exponent β for small values of the probability. Additionally, we have been able to calculate the critical values of the rewiring probability as a function of the number of initial nearest neighbor in small-world networks.

Discontinuous transition to turbulence and nontrivial collective behavior in the turbulent regime are characteristic features of globally coupled maps [10]. These same collective properties emerge in small-world networks as their rewiring probability is increased. Because of the ubiquity of small-world networks in nature and in human-made structures, we may expect to see nontrivial collective behaviors arising in many systems if the appropriate values of relevant parameters are reached. We may expect that other typical phenomena of globally coupled systems, such as the formation of clusters of synchronized elements [19], could also be observed in small-world networks.

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